Heteroskedasticity: *Quick and Dirty One Pager*

OLS: LUEs and BLUEs

1. OLS estimators: Linear Unbiased Estimators (LUEs) and BLUEs (minimum variance in the class of LUE's)
    a. SLR.1-.4 and MLR.1-.4: OLS estimators are LUEs … but please don't celebrate!... as there are lots of linear and unbiased estimators
    b. Adding in SLR.5 and MLR.5 gets us to BLUE… but without, OLS is no longer BLUE!

Learning from the *Sample Mean*

2. Under homoskedasticity:
    a. Linear Unbiased Estimator: $W = \beta_1 Y_1 + \beta_2 Y_2 + ... + \beta_n Y_n$, where $\sum_{i=1}^{n} \beta_i = 1$
    b. BLUE: min $Var(W) = \sum \beta_i^{\,2} \sigma^2 = \sigma^2 \sum \beta_i^{\,2}$ subject to $\sum_{i=1}^{n} \beta_i = 1$ which yields $W = \frac{1}{n} \sum Y_i = \overline{Y}$
3. But with heteroskedasticiy, we now have:
    a. BLUE: min $Var(W) = \sum \beta_i^{\,2} \sigma_i^2$ subject to $\sum_{i=1}^{n} \beta_i = 1$, which yields $\frac{\partial Var(W)}{\partial \beta_i} = \frac{\partial Var(W)}{\partial \beta_j} \Rightarrow 2\sigma_i^{\,2} \beta_i = 2\sigma_j^{\,2} \beta_j$ and $\frac{\beta_i}{\beta_j} = \frac{\sigma_j^{\,2}}{\sigma_i^{\,2}}$, or $\beta_i = \frac{(1/\sigma_i^{\,2})}{\sum_j 1/\sigma_j^{\,2}}$
    b. And so the BLUE will be a weighted average of the sampled values, where the weights are proportional to the inverse of the respective variances.
    c. Intuition: Observations from distributions having larger variances are less reliable, so while you don't want to completely ignore them, you do want to pay them less attention than more reliable observations from distributions with smaller variances. Or put differently: pay less attention to noisier information.

Turning to the SLR Model

4. Under homoskedasticity:
    a. Linear Unbiased Estimator: $W = b_1 Y_1 + b_2 Y_2 + ... + b_n Y_n$, where $\sum b_i = 0$ and $\sum b_i x_i = 1$.
    b. BLUE: min $Var\left[\sum b_i Y_i\right] = \sum b_i^2 \sigma^2$ subject to $\sum b_i = 0$ and $\sum b_i x_i = 1$, which yields $\hat{\beta}_1 = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2} = \frac{S_{xy}}{S_{xx}} = \rho_{xy} \frac{S_y}{S_x}$, and $\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$.

5. But with heteroskedasticiy, we now have:
   a. BLUE: min $Var\left[\sum b_i Y_i\right] = \sum b_i^2 \sigma_i^2$ subject to $\sum b_i = 0$ and $\sum b_i x_i = 1$ … and we can not pull σ^2 out of the summation
   b. Equivalent to min $wgtSSR = \sum \frac{1}{\sigma_i^2}(y_i - \beta_0 - \beta_1 x_i)^2$ … so called ***Weighted Least Squares (WLS)***… and so, and as you saw in the previous example, we weight each observation by the inverse of the variance of that observation… and the same intuition applies: pay less attention to noisier information!
   c. One approach divide $y_i, 1, and\ x_i$ by σ_i and run OLS, since $\sum(\frac{y_i}{\sigma_i} - \beta_0 \frac{1}{\sigma_i} - \beta_1 \frac{x_i}{\sigma_i})^2 = \sum \frac{1}{\sigma_i^2}(y_i - \beta_0 - \beta_1 x_i)^2$
   d. We no longer have $E(MSE) = E\left(\frac{SSR}{n-2}\right) = \sigma^2$ or $Var(B_1) = \frac{\sigma^2}{\sum(x_i - \overline{x})^2}$, since, among other reasons, the σ_i^2 are no longer constant across observations. And so you'd never want to use $\frac{RMSE}{\sqrt{\sum(x_i - \overline{x})^2}}$ (the reported OLS standard error) as the standard error of the estimated slope coefficient. The reported OLS standard errors are no longer relevant… ditto the t stats, p values and Confidence Intervals! ***Say goodbye to inference! … or maybe not?***

So: ***What's a researcher to do?***

6. **Don't despair!**
   a. Run weighted least squares if you can… and if you can't weight by $1/\sigma_i^2$ (who knows the σ_i^2's anyway?), maybe you can use a proxy. And in any event, you might just see if any of this weighting business really matters much (compare to OLS).
   b. ***Robust standard errors***: Add **, robust** to your regression command and generate ***robust*** (*corrected*) standard errors… and as well, *corrected* t stats, p values and Confidence Intervals. *And move on!*
      i. It's OK to add ***, robust*** even if you may not have an issue with Heteroskedasticity. There's no harm in doing this… and besides, you can brag about having ***robust*** standard errors!

7. And in your copious spare time: Read the longer, more detailed and complete, Heteroskedasticity handout to better understand the issues and the remedies. After all, this is just a one… err, make that two, pager!